

# Shape dependence of the finite-size scaling limit in a strongly anisotropic $O(\infty)$ model

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**Abstract.** We discuss the shape dependence of the finite-size scaling limit in a strongly anisotropic  $O(N)$  model in the large- $N$  limit. We show that scaling is observed even if an incorrect value for the anisotropy exponent is considered. However, the related exponents may only be effective ones, differing from the correct critical exponents of the model. We discuss the implications of our results for numerical finite-size scaling studies of strongly anisotropic systems.

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## 1 Introduction

Finite-size scaling (FSS) [1–5] is a very powerful tool that allows to extract information on the critical behavior of a system—which in principle can only be observed in the infinite-volume limit—from finite-volume results. In particular, the most recent Monte Carlo studies heavily rely on FSS for the determination of critical properties (see, *e.g.*, Refs. [6–14] for recent applications to the  $N$ -vector model in two and three dimensions; the list is of course far from being exhaustive). For isotropic systems the recipe is well known. For instance, in two dimensions one considers a domain  $M \times L$  and the finite-size scaling limit  $\xi_{M,L} \rightarrow \infty$ ,  $M, L \rightarrow \infty$  at fixed  $\xi_{M,L}/L$  and aspect ratio  $M/L$ . Here  $\xi_{M,L}$  is a suitably defined finite-volume correlation length. In this limit, long-distance quantities, *e.g.* the susceptibility, show a scaling behavior. For instance, if  $\mathcal{O}$  diverges in the thermodynamic limit as  $|t|^{-x_{\mathcal{O}}}$  for  $t \equiv T - T_c \rightarrow 0$  ( $T$  is the temperature and  $T_c$  its critical value), then in the FSS limit one finds

$$\mathcal{O}(t, M, L) \approx L^{x_{\mathcal{O}}/\nu} f_{\mathcal{O}}(\xi_{M,L}(t)/L, M/L), \quad (1)$$

where  $f_{\mathcal{O}}(x, y)$  is a universal function.

One may also ask what happens if one considers a different limit:  $\xi_{M,L} \rightarrow \infty$ ,  $M, L \rightarrow \infty$  keeping fixed the

ratios  $\xi_{M,L}/L$  and  $M/L^{1+\delta}$  with  $\delta \neq 0$ . If  $\delta > 0$ ,  $M$  increases faster than  $L$  and it is easy to guess that we will obtain an effective strip geometry so that

$$\mathcal{O}(t, M, L) \approx L^{x_{\mathcal{O}}/\nu} f_{\mathcal{O}}(\xi_{M,L}(t)/L, \infty). \quad (2)$$

On the other hand, if  $\delta < 0$ ,  $M$  increases slowly, and, if we are able to keep  $\xi_{M,L}/L$  fixed, *i.e.* to use  $L$  as reference box size (it is not obvious that this is possible), the domain effectively shrinks and becomes one-dimensional. Then, the question is whether a scaling behavior is still observed. One may imagine that the scaling function to be used is that of a one-dimensional system, but in this case it is unclear which exponent should be used in the prefactor.

This problem may appear academic at first but it is motivated by systems that are strongly anisotropic. In such systems correlations increase with exponents that depend on the direction<sup>1</sup>. For instance, the critical two-point function on a lattice  $M \times L$  scales as

$$G(k_{\parallel}, k_{\perp}; M, L) = \lambda^{-2+\eta} G(\lambda^{1+\Delta} k_{\parallel}, \lambda k_{\perp}; \lambda^{-1-\Delta} M, \lambda^{-1} L), \quad (3)$$

where  $k_{\parallel}$  (resp.  $k_{\perp}$ ) is the momentum in the direction of extent  $M$  (resp.  $L$ ) and  $\Delta$  is a new exponent that is usually

<sup>1</sup> The reader should not confuse this class of systems with those that show weak anisotropy. In this case, metric factors but not exponents depend on the direction.

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called *anisotropy* exponent<sup>2</sup>. Because of the anisotropy one must define different finite-volume correlation lengths that measure the correlations in the two directions,  $\xi_{\parallel,M,L}$  and  $\xi_{\perp,M,L}$ . The *canonical* FSS limit is obtained by taking  $\xi_{\perp,M,L} \rightarrow \infty$ ,  $\xi_{\parallel,M,L} \rightarrow \infty$ ,  $M, L \rightarrow \infty$  keeping fixed  $\xi_{\perp,M,L}/L$ ,  $\xi_{\parallel,M,L}/M$ , and the aspect ratio  $M/L^{1+\Delta}$ . Examples of such systems are provided by driven systems that admit nonequilibrium stationary states with strong anisotropy [15,16], surface-growth processes [17], Lifshitz points [18–21], uniaxial magnets with dipolar interactions [22–24], just to mention a few of them. Anisotropic FSS is also relevant for  $d$ -dimensional quantum systems. Indeed, quantum models can be mapped into  $(d+1)$ -dimensional classical systems with strong anisotropy. In this case  $1 + \Delta$  is related to the dynamical critical exponent  $z$  that controls temporal correlations in the classical system [25]. A general approach to scale invariance in infinite volume for these anisotropic systems has been developed by Henkel [26]. However, FSS is still poorly understood. An exact computation on a dimer model which undergoes an anisotropic phase transition was performed by Bhattacharjee and Nagle [27], while, in the context of the FSS for driven diffusive systems, a phenomenological FSS theory was proposed by Binder and Wang [28] and by Leung [29,30], who also gave some heuristic arguments on the consequences of taking the FSS limit with an incorrect anisotropy exponent. Symmetry properties of universal FSS functions in strongly anisotropic systems are discussed in reference [23].

Often in experimental or Monte Carlo applications the exponent  $\Delta$  is not known *a priori* (for instance in nonequilibrium driven systems it cannot be derived directly from the microscopic Hamiltonian [15]), and thus two questions naturally arise. First, if we consider the FSS limit with fixed  $M/L^{1+\Delta+\delta}$ ,  $\delta \neq 0$ , do we still observe scaling? and if yes, with which exponents? Second, how do we determine  $\Delta$ ? An answer to these two questions is of utmost practical importance.

In this work we analyze the finite-size scaling behavior of the  $N$ -vector model in the  $N \rightarrow \infty$  limit, since in this case it is exactly solvable, see, for instance, reference [31]. The FSS behavior has been determined exactly both in the case of short-range [32,33] and of long-range [34–36] interactions. Here, we consider a general class of  $O(\infty)$  models which includes the classical isotropic short-range and long-range cases but also models in which the spin-spin coupling decays with different power-laws in different lattice directions, giving rise to a strongly anisotropic phase transition. For these models we analyze the FSS limit using finite boxes with arbitrary shape.

We show that  $\Delta$  is uniquely determined if one properly measures the correlation length. On the other hand, if only zero-momentum quantities are available, for instance the susceptibility  $\chi$ ,  $\Delta$  cannot be easily determined. Indeed, even if the aspect ratio is incorrect, *i.e.*  $\delta \neq 0$ , one still observes scaling. In some cases, it is even possible to

observe two different FSS limits with the *same* data, one corresponding to the layer/strip geometry, the second one corresponding to a lower-dimensional system.

Our analysis will be limited to systems below the upper critical dimension and above the lower critical one so that we do not consider the problem of anomalous scaling. Also, we will not address the question of the FSS limit at fixed vanishing magnetization, which is of relevance for lattice-gas studies and has already been discussed for isotropic systems in reference [37].

The paper is organized as follows. In Section 2 we define the model, the basic observables, and discuss the large- $N$  limit. In Section 3 we derive the anisotropy exponent  $\Delta$  that defines the aspect ratio and report the FSS functions for the susceptibility and the correlation lengths. In Section 4 we discuss the noncanonical FSS limit in which the ratio  $M/L^{1+\Delta+\delta}$ ,  $\delta \neq 0$ , is kept fixed as the size of the lattice is increased. The results of these two sections are derived in the Appendices. In Section 5 we present a simple numerical example: we consider the standard isotropic model with short-range interactions on a cubic lattice  $M \times L^2$  and show that *two* different scaling behaviors can be observed by keeping fixed the ratio  $M/L^{3/2}$ , in agreement with the theoretical results of Section 4. Finally, in Section 6 we present our conclusions and discuss the implications for numerical studies. In particular, we discuss how one can determine numerically the anisotropy exponent  $\Delta$ .

## 2 The model

We consider a  $d$ -dimensional hypercubic lattice  $\mathbb{Z}^d$ , unit  $N$ -vector spins  $\boldsymbol{\sigma}$  defined at the sites of the lattice, and the Hamiltonian

$$\mathcal{H} = -N \sum_{\mathbf{x}, \mathbf{y}} J(\mathbf{x} - \mathbf{y}) \boldsymbol{\sigma}_{\mathbf{x}} \cdot \boldsymbol{\sigma}_{\mathbf{y}} - Nh \sum_{\mathbf{x}} \sigma_{\mathbf{x}}^1. \quad (4)$$

The partition function is simply

$$Z = \int \prod_{\mathbf{x}} [d\boldsymbol{\sigma}_{\mathbf{x}} \delta(\boldsymbol{\sigma}_{\mathbf{x}}^2 - 1)] e^{-\beta \mathcal{H}}, \quad (5)$$

with  $\beta \equiv 1/T$ . We will be interested in studying the finite-size behavior of the theory. For this purpose, we consider a finite box  $\Lambda_V$  of finite extent  $M$  in the first  $q$  directions (called the “parallel” directions and denoted by the subscript  $\parallel$ ) and  $L$  in the remaining  $p$  directions (called “transverse” and denoted by  $\perp$ ), with  $d = q + p$ , and therefore of volume  $V = M^q L^p$ . In order to be able to consider long-range interactions, we define a finite-size coupling  $J_{M,L}(\mathbf{x})$  as

$$J_{M,L}(\mathbf{x}) = \sum_{\mathbf{n}_{\parallel} \in \mathbb{Z}^q} \sum_{\mathbf{n}_{\perp} \in \mathbb{Z}^p} J(\mathbf{x} + \mathbf{n}_{\parallel} M + \mathbf{n}_{\perp} L), \quad (6)$$

and the finite-size Hamiltonian

$$\mathcal{H} = -N \sum_{\mathbf{x}, \mathbf{y} \in \Lambda_V} J_{M,L}(\mathbf{x} - \mathbf{y}) \boldsymbol{\sigma}_{\mathbf{x}} \cdot \boldsymbol{\sigma}_{\mathbf{y}} - Nh \sum_{\mathbf{x} \in \Lambda_V} \sigma_{\mathbf{x}}^1, \quad (7)$$

<sup>2</sup> The anisotropy exponent should not be confused with the gap exponent  $\Delta \equiv \beta + \gamma$  or with the correction-to-scaling exponent  $\Delta \equiv \nu \nu$  that are usually indicated with the same symbol.

with periodic boundary conditions. Note that definition (6) implies for the Fourier transforms (with  $\mathbf{p} \in \Lambda_V^*$ , see Eq. (11) below)

$$\begin{aligned}\widehat{J}_{M,L}(\mathbf{p}) &= \sum_{\mathbf{x} \in \Lambda_V} e^{i\mathbf{p}\cdot\mathbf{x}} J_{M,L}(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{p}\cdot\mathbf{x}} J(\mathbf{x}) = \widehat{J}(\mathbf{p}).\end{aligned}\quad (8)$$

We consider anisotropic long-range interactions and thus we assume that asymptotically  $\widehat{J}(\mathbf{q})$  has the form

$$\widehat{J}(\mathbf{q}) \simeq \widehat{J}(\mathbf{0}) + a_{\perp} |\mathbf{q}_{\perp}|^{2\rho} + a_{\parallel} |\mathbf{q}_{\parallel}|^{2\sigma}, \quad \text{for } |\mathbf{q}| \rightarrow 0, \quad (9)$$

with  $0 < \rho, \sigma \leq 1$  and  $a_{\perp} < 0$ ,  $a_{\parallel} < 0$  in order to have a ferromagnetic system. For simplicity, we will assume the two metric factors to be equal, and, by redefining the inverse temperature  $\beta$ , we can set  $a_{\perp} = a_{\parallel} = -1/2$ . The case  $0 < \rho < 1$  and  $\sigma = 1$  has been considered in reference [38] in relation to the Casimir amplitudes for long-range quantum systems.

In the large- $N$  limit, assuming periodic boundary conditions and  $h = 0$ , the theory is solved in terms of the gap equations

$$\begin{aligned}\lambda_V \sigma_V &= 0, \\ \beta &= \beta \sigma_V^2 + \frac{1}{L^p M^q} \sum_{\mathbf{q} \in \Lambda_V^*} \frac{1}{K(\mathbf{q}) + \lambda_V},\end{aligned}\quad (10)$$

where  $K(\mathbf{q}) = -2(\widehat{J}(\mathbf{q}) - \widehat{J}(\mathbf{0}))$  and  $\Lambda_V^*$  is the lattice

$$\Lambda_V^* = (2\pi M^{-1} \mathbb{Z}_M^q, 2\pi L^{-1} \mathbb{Z}_L^p). \quad (11)$$

In infinite volume, the same equations hold, with the simple substitution of the summation with the normalized integral over the first Brillouin zone  $[-\pi, \pi]^d$ .

The meaning of the parameters  $\lambda_V$  and  $\sigma_V$  is clarified by considering the magnetization and the two-point function. If  $\langle \cdot \rangle_V$  is the mean value for a system of volume  $V$ , we define

$$M_V = \langle \sigma^1 \rangle_V, \quad G_V(\mathbf{x}) = \langle \sigma_{\mathbf{0}} \cdot \sigma_{\mathbf{x}} \rangle_V. \quad (12)$$

Then

$$M_V = \sigma_V, \quad \widehat{G}_V(\mathbf{q}) = \frac{\beta^{-1}}{K(\mathbf{q}) + \lambda_V}, \quad (13)$$

where  $\widehat{G}(\mathbf{q})$  is the Fourier transform of  $G(\mathbf{x})$ . As we show in Appendix B in the limit  $V \rightarrow \infty$  and in the scaling limit  $\lambda_V \rightarrow 0$ , the correlation function has the form

$$G_{\infty}(\mathbf{x}) = \xi_{\perp, \infty}^{\rho(2-D)} \widetilde{G}_{\infty}(\mathbf{x}_{\perp}/\xi_{\perp, \infty}, \mathbf{x}_{\parallel}/\xi_{\parallel, \infty}), \quad (14)$$

where

$$\xi_{\perp, \infty} = \lambda_{\infty}^{-1/2\rho}, \quad \xi_{\parallel, \infty} = \lambda_{\infty}^{-1/2\sigma}. \quad (15)$$

Therefore, even if the model has long-range correlations in the high- $T$  phase, it is sensible to look at  $\xi_{\perp, \infty}$  and  $\xi_{\parallel, \infty}$  as appropriate typical length scales of the system. We will

refer to them respectively as transverse and longitudinal correlation lengths. They are related by

$$\xi_{\parallel, \infty} = \xi_{\perp, \infty}^{\rho/\sigma}. \quad (16)$$

The critical point is characterized by a vanishing mass gap, *i.e.*  $\xi_{\perp, \infty}^{-1} = \xi_{\parallel, \infty}^{-1} = 0$ , and by a vanishing magnetization,  $\sigma_{\infty} = 0$ , so that the critical temperature is given by (see Eq. (10))

$$\beta_c = \int_{[-\pi, \pi]^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{K(\mathbf{p})}, \quad (17)$$

which is finite whenever the effective dimensionality  $D \equiv p/\rho + q/\sigma$  is greater than 2 and infinite for  $D \leq 2$  (the system undergoes a zero-temperature phase transition). Moreover, given that we will be interested in finite-volume properties, we have  $M_V = 0$  for all values of  $\beta$ , so that we can set  $\sigma_V = 0$  in the gap equation.

### 3 Canonical finite-size scaling

We do not address here the problem of the definition of a finite-volume correlation length and we simply use

$$\xi_{\perp, V} = \lambda_V^{-1/2\rho}, \quad \xi_{\parallel, V} = \lambda_V^{-1/2\sigma}, \quad (18)$$

which are the finite-volume analogues of the infinite-volume correlation lengths defined above.

In order to perform the FSS limit we must identify the correct aspect ratio. From equation (9) we see that  $q_{\perp}^{\rho} \sim q_{\parallel}^{\sigma}$ , *i.e.*  $L^{-\rho} \sim M^{-\sigma}$ . Therefore, the FSS limit should be taken keeping  $S \equiv M/L^{1+\Delta}$ ,  $\Delta \equiv \rho/\sigma - 1$ , fixed. In Appendix A we show that, in the limit  $L, M, \xi \rightarrow \infty$  with  $S$  and  $\xi_{\perp, V}/L$  (or, equivalently,  $\xi_{\parallel, V}/M$ ) fixed, below the upper critical dimension, *i.e.* for  $D < 4$ , the gap equation takes a scaling form. Above the lower critical dimension, *i.e.* for  $D > 2$ , we obtain

$$(4\pi)^{\rho(2-D)/2} (\beta - \beta_c) L^{\rho(D-2)} z^{2-D} = -A_{p,q,\rho,\sigma} + I_{p,q,\rho,\sigma}(z, S), \quad (19)$$

where  $z \equiv (4\pi)^{-\rho/2} (L/\xi_{\perp, V})^{\rho}$ ,  $\beta_c$  is given in equation (17),

$$\begin{aligned}A_{p,q,\rho,\sigma} &\equiv -\frac{1}{(4\pi)^{(p+q)/2}} \frac{1}{\rho\sigma} \\ &\times \frac{\Gamma(p/2\rho)\Gamma(q/2\sigma)}{\Gamma(p/2)\Gamma(q/2)} \Gamma\left(1 - \frac{p}{2\rho} - \frac{q}{2\sigma}\right),\end{aligned}\quad (20)$$

and  $I_{p,q,\rho,\sigma}(z, S)$  is defined in Appendix A, *cf.* equation (60).

For  $D < 2$  there is no finite-temperature phase transition and the gap equation assumes the form

$$(4\pi)^{\rho(2-D)/2} \beta L^{\rho(D-2)} z^{2-D} = -A_{p,q,\rho,\sigma} + I_{p,q,\rho,\sigma}(z, S). \quad (21)$$

Considering the case  $D > 2$ , equation (19) shows that

$$\begin{aligned} \frac{\xi_{\perp,V}}{L} &= f_{\xi,1} \left[ (\beta - \beta_c) L^{\rho(D-2)}, S \right] \\ &= f_{\xi,2} \left[ (\beta - \beta_c) M^{\sigma(D-2)}, S \right], \end{aligned} \quad (22)$$

that allows us to define a ‘‘parallel’’ exponent  $\nu_{\parallel}$  and a ‘‘perpendicular’’ exponent  $\nu_{\perp}$  as

$$\begin{aligned} \nu_{\perp} &= \frac{1}{\rho(D-2)}, & \nu_{\parallel} &= \frac{1}{\sigma(D-2)}, \\ \frac{\nu_{\parallel}}{\nu_{\perp}} &= 1 + \Delta = \frac{\rho}{\sigma}. \end{aligned} \quad (23)$$

Analogously, for the finite-volume susceptibility  $\chi_V = \beta \widehat{G}_V(0)$  we obtain

$$\begin{aligned} \chi_V &= \lambda_V^{-1} = L^{2\rho} f_{\chi,1} \left[ (\beta - \beta_c) L^{\rho(D-2)}, S \right] \\ &= M^{2\sigma} f_{\chi,2} \left[ (\beta - \beta_c) M^{\sigma(D-2)}, S \right]. \end{aligned} \quad (24)$$

The corresponding critical exponent  $\gamma$  associated with the behavior in infinite volume,  $\chi_{\infty} \sim (\beta_c - \beta)^{-\gamma}$ , is correctly identified as  $\gamma = 2\rho\nu_{\perp} = 2\sigma\nu_{\parallel} = 2(D-2)^{-1}$ .

## 4 Noncanonical finite-size scaling

### 4.1 General considerations

In Section 3 we discussed the canonical FSS, obtained by keeping  $S$  constant. However, in many anisotropic systems the correct exponent  $\Delta$  is not known and thus an important question is what happens if we consider the FSS limit keeping fixed the ratio

$$S_{\delta} \equiv \frac{M}{L^{1+\Delta+\delta}} \quad (25)$$

with  $\delta \neq 0$ . If one has separately defined parallel and transverse correlation lengths, one can immediately identify the correct anisotropy exponent  $\Delta$ . It corresponds to the value for which  $\xi_{\perp,V}/L$  and  $\xi_{\parallel,V}/M$  both remain finite in the FSS limit. This uniquely defines the correct exponent. This is not surprising, since the correlation lengths satisfy equation (16) that essentially defines the correct aspect ratio. On the other hand, one may not have access to the correlation lengths but only to some zero-momentum correlation function, for instance to the susceptibility. Thus, one may ask what kind of scaling behavior, if any, is observed if  $S_{\delta}$  is kept fixed. In other words, we may ask if we can find exponents  $\tilde{\gamma}$ ,  $\tilde{\nu}_{\perp}$ , and  $\tilde{\nu}_{\parallel}$  so that asymptotically, for  $L$  and  $M$  going to infinity and  $\beta \rightarrow \beta_c$ , we observe a scaling behavior analogous to that defined in equation (24), *i.e.*

$$\chi = L^{\tilde{\gamma}/\tilde{\nu}_{\perp}} f_{\chi,1} \left[ (\beta - \beta_c) L^{1/\tilde{\nu}_{\perp}}, S_{\delta} \right] \quad (26)$$

$$= M^{\tilde{\gamma}/\tilde{\nu}_{\parallel}} f_{\chi,2} \left[ (\beta - \beta_c) M^{1/\tilde{\nu}_{\parallel}}, S_{\delta} \right]. \quad (27)$$

Of course, the second important question is the relation of these exponents with the correct ones defined above. Again, we will restrict our attention to the case  $2 < D < 4$ , *i.e.* below the upper critical dimension and above the lower critical one.

In the following, we only consider the case  $\delta > 0$ , given that the case  $-1 - \Delta < \delta < 0$  is obtained by performing the substitutions

$$\begin{aligned} p &\leftrightarrow q, & L &\leftrightarrow M, & \rho &\leftrightarrow \sigma, \\ \delta &\mapsto -\delta\sigma^2/\rho(\rho + \sigma\delta), & S_{\delta} &\mapsto S_{\delta}^{-1/(\rho/\sigma + \delta)}. \end{aligned} \quad (28)$$

If  $\delta$  is positive,  $S$  goes to infinity as  $L, M \rightarrow \infty$ . We must then study the gap equation in that limit, requiring at the same time  $\lambda_V \rightarrow 0$  in an arbitrary way. The detailed calculation is presented in Appendix C, where we show that there are two nontrivial cases:

- One can take the limit  $S \rightarrow \infty$  at  $z^2 \sim \lambda_V L^{2\rho}$  fixed. In terms of the correlation lengths this corresponds to the FSS limit at  $\xi_{\perp,V}/L$  fixed. At the same time  $\xi_{\parallel,V}/M \sim (\xi_{\perp,V}/L)^{\rho/\sigma} L^{-\delta}$  goes to zero, *i.e.*  $M$  increases faster than parallel correlations. Therefore, the final ‘‘effective’’ geometry is  $\infty^q \times L^p$ , so that we obtain the FSS behavior of a ‘‘layer’’.
- Alternatively, one can take the limit  $S \rightarrow \infty$ , with  $z \rightarrow 0$ , keeping  $z^2 S^{2\sigma} \sim \lambda_V M^{2\sigma}$  fixed. This corresponds to keeping fixed the ratio  $\xi_{\parallel,V}/M$  while at the same time  $\xi_{\perp,V}/L$  diverges. The expected ‘‘effective’’ geometry should be that of a  $q$ -dimensional hypercube with linear size  $M$  ( $M^q \times 0^p$ ).

The first limit always exists and it is easy to predict the result. It correspond to the standard FSS limit for a system  $\infty^q \times L^p$  and thus the scaling in terms of  $L$  is canonical. Therefore, it corresponds to taking  $\beta \rightarrow \beta_c$  and  $L \rightarrow \infty$  keeping fixed  $(\beta - \beta_c) L^{\rho(D-2)}$ , which is indeed the correct combination. Then, we obtain

$$\chi = L^{2\rho} f_{\chi,1} \left[ (\beta - \beta_c) L^{\rho(D-2)}, \infty \right] \quad (29)$$

and, using equation (25),

$$\chi = \left( \frac{M}{S_{\delta}} \right)^{\frac{2\rho}{1+\Delta+\delta}} f_{\chi,1} \left[ (\beta - \beta_c) \left( \frac{M}{S_{\delta}} \right)^{\frac{\rho(D-2)}{1+\Delta+\delta}}, \infty \right]. \quad (30)$$

Thus  $\tilde{\nu}_{\perp} = \nu_{\perp}$  and  $\tilde{\gamma} = \gamma$  (note that both equations give the same result for  $\gamma$ ). On the other hand, the parallel exponent  $\tilde{\nu}_{\parallel}$  is given by

$$\tilde{\nu}_{\parallel} = \frac{1 + \Delta + \delta}{1 + \Delta} \nu_{\parallel}, \quad (31)$$

a result which is a trivial consequence of the fact that, if  $S_{\delta}$  is fixed, then  $\tilde{\nu}_{\parallel}/\tilde{\nu}_{\perp} = 1 + \Delta + \delta$ . Therefore, even if  $\delta \neq 0$ , one still observes scaling. Transverse exponents are correct, while longitudinal ones are only effective. Let us finally note that equations (29, 30) give both the correct infinite-volume behavior for  $\chi$ . Indeed, the existence of a finite infinite-volume limit for  $\beta < \beta_c$  implies

$f_{\chi,1}(x) \sim (-x)^{-2/(D-2)} \sim (-x)^{-\gamma}$  for  $x \rightarrow -\infty$  and therefore equations (29, 30) are consistent with  $\chi_\infty(\beta) \sim (\beta_c - \beta)^{-\gamma}$ . In other words, one must have  $\tilde{\gamma} = \gamma$  in order to have the correct infinite-volume limit.

Limit (b) is much less conventional and we will show that such a limit exists (in the sense that  $\chi$  has a scaling behavior) only if  $q/\sigma < 2$ , *i.e.* if the  $q$ -dimensional theory is below the lower critical dimension: in this case however, the exponents obtained from the data collapse do not have anything to do with the correct ones, while the scaling functions depend only on the  $q$ -dimensional theory, *i.e.* theories with different  $p$  and/or  $\rho$ —they are thus physically inequivalent—have the *same* scaling functions. Thus, in this case from the observation of a good scaling behavior one can draw the incorrect conclusion that physically inequivalent theories have the same critical behavior.

At this point the reader may be puzzled by the fact that for the same  $S_\delta$  we observe two different types of scaling. Mathematically this is related to the fact that in the FSS limit we must follow a path in the  $\beta, L$  plane with  $\beta \rightarrow \beta_c$  and  $L \rightarrow \infty$ . The two limits correspond to two different families of paths. Unexpectedly scaling is observed on both of them.

## 4.2 Limit (b)

We wish now to consider the limit  $S \rightarrow \infty, z \rightarrow 0$  at fixed  $zS^\sigma$ . Such a limit has been considered in Appendix C where it is shown that the leading contribution to the gap equation has the form

$$(4\pi)^{\rho(2-D)/2+d/2}(\beta - \beta_c)L^{\rho(D-2)} = (4\pi)^{q/2}r^{\sigma-q/2}\omega^{-2+q/\sigma} [I_{q,\sigma}^{\text{iso}}(\omega) - A_{q,\sigma}] + K, \quad (32)$$

where

$$r \equiv (4\pi)^{\rho/\sigma-1}S^2, \\ \omega \equiv zr^{\sigma/2} = (4\pi)^{-\sigma/2}\lambda_V^{1/2}M^\sigma, \quad (33)$$

$A_{q,\sigma} \equiv A_{0,q,\rho,\sigma}$  (it does not depend on  $\rho$ ), and  $I_{q,\sigma}^{\text{iso}}(\omega)$  is the analogue of  $I_{p,q,\rho,\sigma}(z, S)$  appearing in equations (19) and (21) for an isotropic  $q$ -dimensional system with long-range exponent  $\sigma$ . The constant  $K$ , *cf.* equation (88), gives a subleading correction for  $q/\sigma < 2$ . We will now distinguish two cases, depending on the value of  $q/\sigma$ .

### 4.2.1 Case $0 < q/\sigma < 2$

In this case we can neglect  $K$  and set  $r = r_\delta L^{2\delta}$ , where  $r_\delta \equiv (4\pi)^{\rho/\sigma-1}S_\delta^2$  (by definition  $r_0 = r$ ) is kept fixed in the limit. Then, for  $\beta > \beta_c$ , we can rewrite

$$(4\pi)^{\rho(2-D)/2+p/2}(\beta - \beta_c)\Xi = I_{q,\sigma}^{\text{iso}}(\omega) - A_{q,\sigma} \quad (34)$$

with

$$\Xi \equiv r_\delta^{q/2-\sigma} L^{\rho(D-2)-2\delta(\sigma-q/2)} \omega^{2-q/\sigma}.$$

This equation has the same functional form as the gap equation of an isotropic  $q$ -dimensional system with effective dimension  $d_{\text{eff}} = q/\sigma < 2$ , *cf.* equation (21). Therefore, if for a  $q$ -dimensional isotropic system of extent  $M$  and long-range exponent  $\sigma$  we have,

$$\chi^{\text{iso}} = M^{2\sigma} f_\chi^{\text{iso}} [M^{\sigma(d_{\text{eff}}-2)} \beta], \quad (35)$$

then in the limit we are considering we obtain

$$\chi = M^{2\sigma} f_\chi^{\text{iso}} [S_\delta^{-p/(1+\Delta+\delta)} M^{1/\tilde{\nu}_\parallel} (\beta - \beta_c)], \quad (36)$$

where

$$\tilde{\nu}_\parallel^{-1} = \frac{(1 + \Delta)\nu_\parallel^{-1} + \delta\sigma(d_{\text{eff}} - 2)}{1 + \Delta + \delta}. \quad (37)$$

We note that the second term in the numerator of equation (37) is equal to  $\delta\nu_{\parallel,\text{iso}}^{-1}$ , where  $\nu_{\parallel,\text{iso}} = [\sigma(d_{\text{eff}} - 2)]^{-1}$  is the correct exponent for the  $d_{\text{eff}}$ -dimensional isotropic system, so that equation (37) interpolates between  $\nu_\parallel^{-1}$  and  $\nu_{\parallel,\text{iso}}^{-1}$ .

There are two limitations to the validity of equation (36). First, equation (36) holds only in the limit in which the argument is constant as  $M \rightarrow \infty$ . Since we are considering  $\beta \rightarrow \beta_c$ , this requires  $\tilde{\nu}_\parallel > 0$  and in turn

$$\delta < \delta_{\text{max}} = \frac{(D-2)(1+\Delta)}{(2-d_{\text{eff}})}. \quad (38)$$

If  $\delta$  is larger than  $\delta_{\text{max}}$  no scaling is observed by taking  $\beta \rightarrow \beta_c$ . One observes a truly  $q$ -dimensional FSS behavior: a scaling behavior is obtained only by taking  $\beta \rightarrow \infty$ .

Additionally, it should be observed that  $f_\chi^{\text{iso}}(x)$  is defined only for  $x > 0$ , and indeed a solution to equation (34) is found only when the left-hand side is positive. In Appendix C we show that the correct extension for  $x < 0$  is  $f_\chi^{\text{iso}}(x) = 0$  in the sense that, for  $\beta < \beta_c$ , we have  $\chi/M^{2\sigma} \rightarrow 0$ .

We also obtain

$$\tilde{\gamma} = 2\sigma\tilde{\nu}_\parallel, \\ \tilde{\nu}_\perp = \frac{\tilde{\nu}_\parallel}{1 + \Delta + \delta}. \quad (39)$$

The effective exponents are thus unrelated to the correct ones and vary continuously with  $\delta$ , interpolating between the correct ones ( $\delta = 0$ ) and those of a  $d_{\text{eff}}$ -dimensional system ( $\delta = \delta_{\text{max}}$ ).

At this point the reader may be puzzled by the fact that  $\tilde{\gamma} \neq \gamma$ , since we already claimed in Section 4.1 that this is a necessary condition to ensure the correct infinite-volume limit. We now show that there is no contradiction. The effective exponent  $\tilde{\gamma}$  differs from  $\gamma$  because in limit (b) the scaling function  $f_\chi(x)$  vanishes identically in the whole high-temperature phase  $x < 0$ , *i.e.* in the only phase in which  $\chi$  is finite ( $\chi_\infty(\beta) = \infty$  in the whole low-temperature phase). In other words, scaling (b) does not give any information on the infinite-volume behavior, and thus there is no surprise for  $\tilde{\gamma} \neq \gamma$ .

#### 4.2.2 Case $q/\sigma > 2$

In this case  $K$  in equation (32) cannot be neglected, and therefore no scaling solution can be found. Scaling can only be observed along a very specific trajectory in the  $\beta$ ,  $L$  plane. Indeed, if we consider the line

$$\beta_c(L) = \beta_c + aL^{-1/\nu_\perp}, \quad (40)$$

where

$$a = (4\pi)^{-\rho(2-D)/2-d/2} K, \quad (41)$$

we still observe scaling in the sense that asymptotically

$$\chi = S_\delta^{2\sigma} L^{2\sigma(1+\Delta+\delta)} f_\chi^{\text{iso}} \left[ S_\delta^{-2\sigma+q} L^{1/\tilde{\nu}_\perp} (\beta - \beta_c(L)) \right], \quad (42)$$

where  $\tilde{\nu}_\perp^{-1} = \nu_\perp^{-1} + \delta\sigma(d_{\text{eff}} - 2) = \nu_\perp^{-1} + \delta\nu_{\parallel,\text{iso}}^{-1}$  and  $f_\chi^{\text{iso}}$  is the scaling function of a  $d_{\text{eff}}$ -dimensional isotropic system (with  $d_{\text{eff}} = q/\sigma > 2$ ). However, this is not standard scaling since  $a$  must be tuned to a proper value: it is not enough to take  $\beta \rightarrow \beta_c$ , but it is also necessary to take this limit along the line (40).

## 5 A numerical example

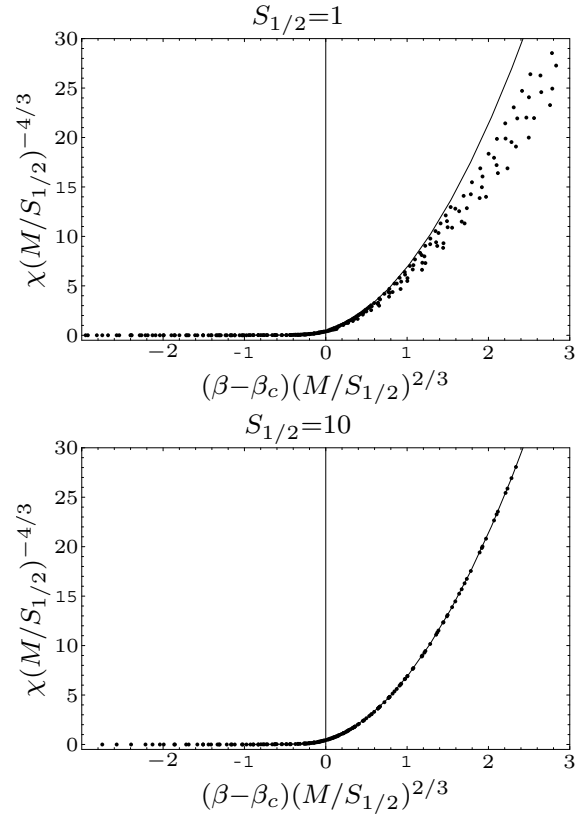
We wish now to present a numerical example, in order to clarify the issues discussed in the previous sections. We consider the standard short-range model with nearest-neighbor couplings, so that  $\rho = \sigma = 1$  and  $\Delta = 0$ . We consider a three-dimensional cubic lattice of dimensions  $M \times L^2$ , *i.e.*  $p = 2$  and  $q = 1$ . For this system  $\beta_c$  can be easily computed to arbitrary precision. Indeed, the integral (17) can be written as [39]

$$\beta_c = \frac{\sqrt{3} - 1}{192\pi^3} \Gamma(1/24)^2 \Gamma(11/24)^2 \approx 0.25273101. \quad (43)$$

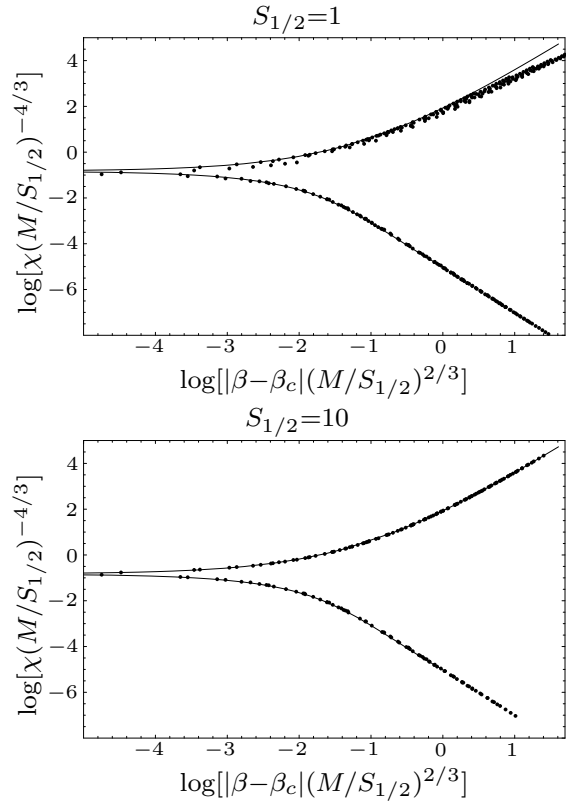
According to the results of Section 4.2 we should be able to observe a noncanonical scaling for  $0 < \delta < 1$ . We thus fix  $\delta = 1/2$  and compute  $\chi$  for several values of  $\beta$ ,  $L$ , and  $M$  corresponding to  $S_{1/2} = 1$  and  $S_{1/2} = 10$ . We use  $5 \leq L \leq 245$  and  $0.235 \leq \beta \leq 0.279$ . The previous results predict two possible scalings. Scaling (a) corresponds to consider  $\chi/M^{4/3}$  versus  $(\beta - \beta_c)M^{2/3}$ , while scaling (b) corresponds to consider  $\chi/M^2$  versus  $(\beta - \beta_c)M^{1/3}$ .

In Figure 1 we report the results for limit (a). For  $S_{1/2} = 1$  there are somewhat large corrections to scaling and the convergence is slow in the low-temperature region. For  $S_{1/2} = 10$  the agreement is very good: no corrections can be seen on this scale. This different behavior is due to the different lattice sizes that are used: the data have the same values of  $L$  and therefore lattices with  $S_{1/2} = 10$  are ten times larger than those with  $S_{1/2} = 1$ . In Figure 2 we show the same data in a log-log plot. Here we see better the results corresponding to the high-temperature phase, although corrections in the low-temperature phase are somewhat less visible.

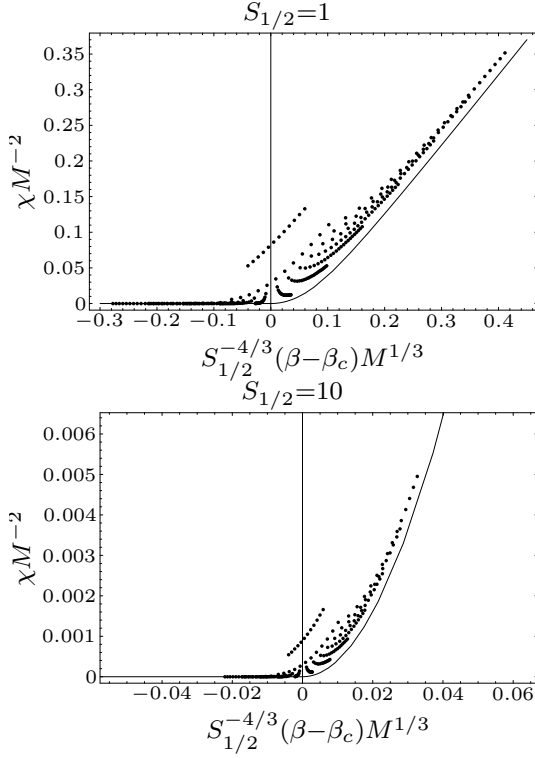
In Figure 3 we report the results for limit (b). The data show a reasonably good scaling behavior, although



**Fig. 1.** Noncanonical FSS scaling in 3 dimensions: limit (a). The solid line is the theoretical prediction. Here  $\delta = 1/2$ .



**Fig. 2.** Noncanonical FSS scaling in 3 dimensions: limit (a). The same data of Figure 1 in a log-log plot.



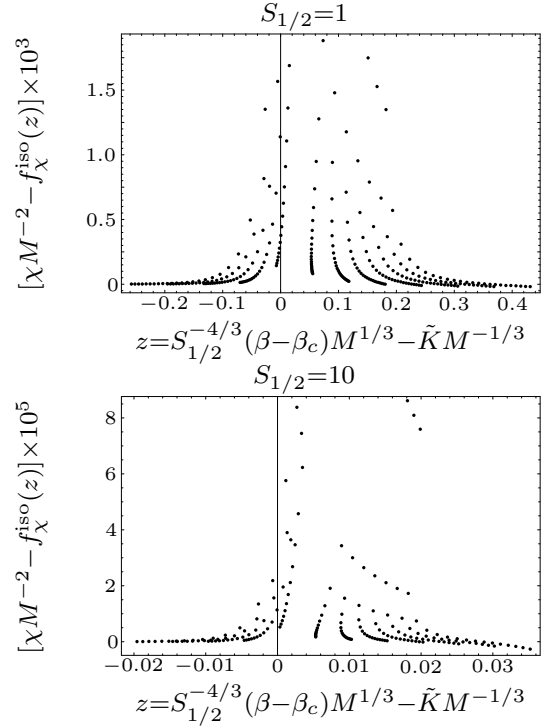
**Fig. 3.** Noncanonical FSS scaling in 3 dimensions: limit (b). The solid line is the theoretical prediction. Here  $\delta = 1/2$ .

the convergence to the limiting curve, corresponding to a one-dimensional system (see Ref. [40] for a discussion of the FSS behavior of one-dimensional  $O(N)$  models), is quite slow. For limit (b) we have also investigated the behavior of the leading correction term. In Appendix C.3 we compute the first scaling correction to equation (36). It can be effectively taken into account by a size-dependent shift of the critical temperature, *i.e.* by writing

$$\chi M^{-2} = f_{\chi}^{\text{iso}} \left[ S_{\delta}^{-4/3} (\beta - \beta_c) M^{1/3} - \tilde{K} M^{-1/3} \right], \quad (44)$$

where  $\tilde{K} \equiv K(4\pi)^{-1} S_{\delta}^{-2/3}$ ,  $K$  is defined in equation (88). Numerically, in this particular case,  $K \approx -3.9002649$ . In Figure 4 we plot the difference between the left-hand and the right-hand side of equation (44) *versus* the argument of the function  $f_{\chi}^{\text{iso}}$ . It is clear that data points in Figure 4 are converging towards zero, confirming that asymptotically  $f_{\chi}^{\text{iso}}(x)$  is indeed the correct scaling function. A numerical analysis shows that corrections to equation (44) vanish as  $M^{-1}$ .

From a practical point of view it is important to be able to distinguish scaling (a) from scaling (b). This is not very difficult since in scaling (b)  $f_{\chi}(x) = 0$  in the region  $x < 0$ . This observation provides a simple criterion: if the high-temperature data scale onto a nontrivial curve we are clearly observing limit (a): for instance the scaling curve appearing in Figure 2 can only correspond to scaling (a).



**Fig. 4.** Corrections to the FSS behavior: limit (b). Here  $\delta = 1/2$ .

## 6 Conclusions

In this paper we have considered an anisotropic  $O(\infty)$  model, focusing on the shape dependence of the FSS limit and in particular on the role of the aspect ratio. Considering for instance the susceptibility, we have shown that, even if the FSS limit is taken by keeping constant an incorrect aspect ratio  $M/L^{1+\Delta+\delta}$ , one still observes scaling, *i.e.* the data satisfy the scaling forms (26) and (27), although the effective exponents differ from the correct ones. Therefore, the numerical observation of data collapse does not give direct information on the correct values of the anisotropy exponent  $\Delta$  and of the critical exponents. In the case of limit (a), if we consider the scaling at fixed  $M/L^{\alpha}$ , we obtain for  $\alpha \geq 1 + \Delta$

$$\begin{aligned} \tilde{\nu}_{\perp} &= \nu_{\perp} \\ \tilde{\nu}_{\parallel} &= \alpha \nu_{\parallel} \\ \tilde{\gamma} &= \gamma, \end{aligned} \quad (45)$$

while for  $\alpha \leq 1 + \Delta$  we have

$$\begin{aligned} \tilde{\nu}_{\perp} &= \nu_{\parallel} / \alpha \\ \tilde{\nu}_{\parallel} &= \nu_{\parallel} \\ \tilde{\gamma} &= \gamma. \end{aligned} \quad (46)$$

These expressions have been explicitly verified in the specific model we consider, but we expect them to be valid for generic anisotropic systems. Indeed, if for instance  $\alpha > 1 + \Delta$ , we always expect

$$\chi = L^{\gamma/\nu_{\perp}} f \left[ (\beta - \beta_c) L^{1/\nu_{\perp}}, \infty \right] \quad (47)$$

and, by simply using the fact that  $S_\alpha \equiv ML^{-\alpha}$  is constant, we obtain

$$\chi = \left(\frac{M}{S_\alpha}\right)^{\gamma/(\alpha\nu_\perp)} f\left[(\beta - \beta_c) \left(\frac{M}{S_\alpha}\right)^{1/(\alpha\nu_\perp)}, \infty\right], \quad (48)$$

that gives equation (45). Note that this simple argument does not only predict the values of the effective critical exponents, but also a very simple dependence on  $S_\alpha$ .

In the case of limit (b) we are not much interested in the specific form of the effective exponents. The only important feature that we must notice is that all exponents vary continuously with  $\alpha$  and, if  $\alpha$  is sufficiently different from  $1 + \Delta$ , scaling is observed only for  $\beta \rightarrow \infty$ .

For numerical applications, it is very important to establish a strategy that can be used to correctly identify the exponents in generic anisotropic systems. One possibility consists in performing a numerical study keeping fixed  $M/L^\alpha$  for different values of  $\alpha$ . If one observes that the data obtained with two different values of  $\alpha$  collapse onto a single curve by using the *same* exponents and the *same* value of  $\beta_c$ , then the exponents which are determined in this way are correct. Indeed, this implies that one is observing limit (a)—in limit (b) all exponents vary continuously—and therefore equations (45, 46) apply. Note that as an additional check one can see whether  $f_\chi(x)$  is nontrivial for  $x < 0$ : if this the case, one is observing limit (a). To be more specific, suppose that the scaling behavior (26) is observed for  $\alpha_1$  and  $\alpha_2$ ,  $\tilde{\gamma}$  and  $\tilde{\nu}_\perp$  being the same in both cases; then  $\tilde{\gamma}$  and  $\tilde{\nu}_\perp$  can be identified with  $\gamma$  and  $\nu_\perp$  and  $1 + \Delta < \min(\alpha_1, \alpha_2)$ . From the data at  $\alpha_1$  and  $\alpha_2$  one cannot determine  $\nu_\parallel$  and  $\Delta$ . For this purpose, one must perform simulations at a value  $\alpha_3$  such that  $\tilde{\nu}_\perp \neq \nu_\perp$ , *i.e.* one must find a value such that  $\alpha_3 < 1 + \Delta$ . Then, one should use equation (27) setting  $\tilde{\gamma} = \gamma$  (this is meant to avoid limit (b)) and using  $\tilde{\nu}_\parallel$  as a free parameter. Then  $\nu_\parallel = \tilde{\nu}_\parallel$  and  $1 + \Delta = \nu_\parallel/\nu_\perp$ . As a check, one can analyze the data at  $\alpha_1$  and  $\alpha_2$  and look for the scaling behavior (27), fixing  $\gamma$  to avoid limit (b). The effective exponents  $\tilde{\nu}_\parallel$  should satisfy  $\tilde{\nu}_\parallel = \alpha\nu_\parallel/(1 + \Delta)$ . Thus, in principle, by studying the behavior for three different values of  $\alpha$  one can determine all exponents. However, in practice this may be difficult, because it requires the ability to distinguish corrections which vanish as  $M, L \rightarrow \infty$  from corrections that persist in the limit  $M, L \rightarrow \infty$  and that require therefore the use of different values of the effective exponents. Note that this method relies only on equations (45, 46) that we expect to be valid in general and not only in our specific model.

Let us note that all the above-discussed ambiguities are not present if the correlation lengths are measured, since their scaling behavior is fixed,  $\xi_\perp \sim L$ ,  $\xi_\parallel \sim M$ . Deviations from these laws give immediately the exponent  $\Delta$ . Therefore, finite-size numerical studies of anisotropic systems should always determine finite-volume correlation lengths, as done for instance in reference [41] for a driven lattice gas.

A somewhat unexpected result of our analysis is that in some cases it is possible to observe two different types of effective scaling: one corresponding to the layer/strip

geometry  $\infty^q \times L^p$ , which is the scaling one would naturally expect, and the other one corresponding to a lower-dimensional system  $M^q \times 0^p$ . The possibility of this type of scaling is not obvious since it requires that one correlation length increases much faster than the shortest of the two lattice sizes. Even less clear is why this limit exists only if  $q$  is below the lower critical dimension of the model (for larger values of  $q$  such a scaling can still be observed by performing an additional tuning). We expect that scaling (b) is strictly related to the fact that  $\chi$  and  $\xi$  as defined here diverge in the whole low-temperature region, but it is difficult to transform this conjecture into a quantitative argument. In Ising systems it is simple to avoid this feature. For instance, one could define a finite-volume susceptibility as

$$\chi = \beta \left( \widehat{G}(0) - VM^2 \right), \quad (49)$$

where

$$M = \frac{1}{V} \left\langle \left| \sum_{i \in \Lambda_V} \sigma_i \right| \right\rangle. \quad (50)$$

An open question is whether such a quantity may show a noncanonical scaling (b). Our guess is that only scaling (a) is possible in this case.

We thank Daniel Dantchev for useful comments.

## Appendix A: Gap equation: Finite-size scaling limit

In this Appendix we wish to determine the asymptotic form of the gap equation for  $\lambda_V \rightarrow 0$ ,  $M, L \rightarrow \infty$ , without making any hypothesis on the way in which the limit is taken. We will assume to be below the upper critical dimension, *i.e.*  $D < 4$  where  $D \equiv p/\rho + q/\sigma$ . We will consider separately the cases  $D < 2$  and  $D > 2$ .

### A.1 Gap equation above the lower critical dimension: $2 < D < 4$

We start by rewriting (see Ref. [33]) the gap equation as

$$\beta - \beta_c = -\lambda_V \int_{[-\pi, \pi]^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{K(\mathbf{p})(K(\mathbf{p}) + \lambda_V)} + \Sigma, \quad (51)$$

where

$$\Sigma \equiv \frac{1}{V} \sum_{\mathbf{p} \in \Lambda_V^*} \frac{1}{K(\mathbf{p}) + \lambda_V} - \int_{[-\pi, \pi]^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{K(\mathbf{p}) + \lambda_V}. \quad (52)$$

Now, for  $D < 4$  and  $\lambda_V \rightarrow 0$  we can expand  $K(\mathbf{p})$  in the integral appearing in equation (51) and extend the integration over all  $\mathbb{R}^d$ , obtaining

$$\beta - \beta_c \approx -\lambda_V^{D/2-1} B_1 + \Sigma, \quad (53)$$



where

$$B_1 \equiv \int_{\mathbb{R}^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{K_c(\mathbf{p})(K_c(\mathbf{p}) + 1)} = A_{p,q,\rho,\sigma}, \quad (54)$$

with  $K_c(\mathbf{p}) = |\mathbf{p}_\perp|^{2\rho} + |\mathbf{p}_\parallel|^{2\sigma}$  and  $A_{p,q,\rho,\sigma}$  defined in equation (20). Using the Poisson summation formula for a periodic function  $f(x) = f(x + l)$ ,

$$\sum_{n=0}^{l-1} f(n) = 2\pi \sum_{k=-\infty}^{\infty} \int_0^l dx f(x) e^{2\pi i k x}, \quad (55)$$

we can rewrite

$$\begin{aligned} \Sigma &= \sum_{\mathbf{n} \neq \mathbf{0}} \int_{[-\pi, \pi]^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{e^{i\mathbf{p}_\parallel \mathbf{n}_\parallel M + i\mathbf{p}_\perp \mathbf{n}_\perp L}}{K(\mathbf{p}) + \lambda_V} \\ &\approx \sum_{\mathbf{n} \neq \mathbf{0}} \int_{\mathbb{R}^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{e^{i\mathbf{p}_\parallel \mathbf{n}_\parallel M + i\mathbf{p}_\perp \mathbf{n}_\perp L}}{K_c(\mathbf{p}) + \lambda_V}, \end{aligned} \quad (56)$$

where in the last step we have assumed  $L, M \rightarrow \infty$ . Convergence is guaranteed by the oscillating phase factor.

In order to compute the asymptotic behavior of  $\Sigma$  for  $\lambda_V \rightarrow 0$ ,  $M, L \rightarrow \infty$ , we introduce the Laplace transform  $f_\alpha(p)$  of  $\exp(-u^\alpha)$ , *i.e.* we define

$$\int_0^\infty dp e^{-up} f_\alpha(p) = e^{-u^\alpha}, \quad (57)$$

for  $0 < \alpha \leq 1$ . Of course, for  $\alpha = 1$ ,  $f_1(p) = \delta(p - 1)$ . It is easy to show that for  $0 < \alpha < 1$

$$f_\alpha(p) \sim p^{-(1+\alpha)}, \quad (58)$$

for  $p \rightarrow +\infty$ , and

$$f_\alpha(p) \sim p^{-\frac{2-\alpha}{2(1-\alpha)}} \exp[-(1-\alpha)(p/\alpha)^{-\frac{\alpha}{1-\alpha}}], \quad (59)$$

for  $p \rightarrow 0+$ . Then, we obtain

$$\begin{aligned} \Sigma &= \sum_{\mathbf{n} \neq \mathbf{0}} \int_{\mathbb{R}^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \\ &\quad \times \int_0^\infty dt e^{-\lambda_V t - t|\mathbf{p}_\perp|^{2\rho} - t|\mathbf{p}_\parallel|^{2\sigma} + i\mathbf{p}_\perp \cdot \mathbf{n}_\perp L + i\mathbf{p}_\parallel \mathbf{n}_\parallel M} \\ &= \sum_{\mathbf{n} \neq \mathbf{0}} \int_0^\infty d\eta d\tau f_\rho(\eta) f_\sigma(\tau) \int_{\mathbb{R}^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \\ &\quad \times \int_0^\infty dt e^{-\lambda_V t - \eta t |\mathbf{p}_\perp|^{2\rho} - \tau t |\mathbf{p}_\parallel|^{2\sigma} + i\mathbf{p}_\perp \cdot \mathbf{n}_\perp L + i\mathbf{p}_\parallel \mathbf{n}_\parallel M} \\ &= \frac{L^{(2-D)\rho}}{(4\pi)^{\rho+(d-\rho D)/2}} \\ &\quad \times \int_0^\infty d\eta d\tau f_\rho(\eta) f_\sigma(\tau) H_{p,q}(\eta, \tau, z, r) \\ &\equiv \lambda_V^{D/2-1} I_{p,q,\rho,\sigma}(z, S), \end{aligned} \quad (60)$$

where

$$\begin{aligned} S &\equiv M/L^{\rho/\sigma}, & r &\equiv (4\pi)^{\rho/\sigma-1} S^2, \\ z^2 &\equiv (4\pi)^{-\rho} L^{2\rho} \lambda_V, \\ \eta_t &\equiv t^{1/\rho} \eta, & \tau_t &\equiv t^{1/\sigma} \tau, \end{aligned} \quad (61)$$

$$\begin{aligned} H_{p,q}(\eta, \tau, z, r) &\equiv \int_0^\infty dt \eta_t^{-p/2} \tau_t^{-q/2} e^{-z^2 t} \\ &\quad \times [B^p(\eta_t^{-1}) B^q(r\tau_t^{-1}) - 1], \end{aligned} \quad (62)$$

and

$$B(s) \equiv \sum_{n \in \mathbb{Z}} e^{-\pi n^2 s}. \quad (63)$$

Note the well-known property

$$B(s) = s^{-1/2} B(1/s), \quad (64)$$

that implies  $B(s) \approx s^{-1/2}$  for  $s \rightarrow 0$ .

Using equations (53) and (60) we obtain equation (19). Note that for an isotropic geometry with  $q = 0$ ,  $I_{p,0,\rho,0}(z, S)$  does not depend on  $S$  and thus we will simply write  $I_{p,\rho}^{\text{iso}}(z) \equiv I_{p,0,\rho,0}(z, S)$ . For a geometry with  $M = \infty$ —we call it layer geometry—equation (19) still holds with  $S = \infty$ . In equation (62) it corresponds to setting  $r = \infty$ , *i.e.* to replacing  $B^q(r\tau_t^{-1})$  with 1.

In the low-temperature phase ( $\beta > \beta_c$  fixed) we have  $\lambda_V \rightarrow 0$  for all  $\beta$  when  $V \rightarrow \infty$ . Thus, we can use the above-reported expressions to determine the infinite-volume behavior of  $\xi_{\perp,V}$ . From equation (53) we obtain  $\Sigma = \beta - \beta_c > 0$  in the infinite-volume limit. Because of the prefactor  $L^{(2-D)\rho}$  appearing in the last term of equation (60), since  $D > 2$ , this is possible only if the integral, *i.e.*  $H_{p,q}(\eta, \tau, z, r)$  diverges (the  $\eta$  and  $\tau$  integrations are completely harmless), which only happens for  $z \rightarrow 0$ . In order to compute the asymptotic behavior of  $H_{p,q}(\eta, \tau, z, r)$  for  $z \rightarrow 0$ , we use the duality property (64) to rewrite

$$\begin{aligned} H_{p,q}(\eta, \tau, z, r) &= \\ &\int_0^1 dt \eta_t^{-p/2} \tau_t^{-q/2} e^{-z^2 t} [B^p(\eta_t^{-1}) B^q(r\tau_t^{-1}) - 1] \\ &\quad + \int_1^\infty dt r^{-q/2} e^{-z^2 t} [B^p(\eta_t) B^q(\tau_t/r) - 1] \\ &\quad + \int_1^\infty dt e^{-z^2 t} (r^{-q/2} - \eta_t^{-p/2} \tau_t^{-q/2}). \end{aligned} \quad (65)$$

The first two integrals are always finite while the third one gives  $H_{p,q}(\eta, \tau, z, r) \approx r^{-q/2}/z^2$ . This implies that in the infinite-volume limit

$$\xi_{\perp,V} \sim S^{q/2\rho} L^{\rho D/2}. \quad (66)$$

## A.2 The case $D < 2$

For  $D < 2$  little changes. Now criticality is observed only for  $\beta \rightarrow \infty$ . We rewrite the gap equation as

$$\beta = \int_{[-\pi, \pi]^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{K(\mathbf{p}) + \lambda_V} + \Sigma. \quad (67)$$

For  $D < 2$  and  $\lambda_V \rightarrow 0$ , we can expand  $K(\mathbf{p})$  in the first term and extend the integration over all  $\mathbb{R}^d$ , while  $\Sigma$  can be treated as before. We obtain

$$\beta = \lambda_V^{D/2-1} B_2 + \lambda_V^{D/2-1} I_{p,q,\rho,\sigma}(z, S), \quad (68)$$

where

$$B_2 \equiv \int_{\mathbb{R}^d} \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{1}{K_c(\mathbf{p}) + 1} = -A_{p,q,\rho,\sigma}, \quad (69)$$

and  $A_{p,q,\rho,\sigma}$  is defined in equation (20). Equation (21) follows immediately.

## Appendix B: FSS of the correlation function

Now we consider the (real-space) correlation function  $G_V(\mathbf{x})$ :

$$G_V(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{p} \in \Lambda_V^*} \frac{e^{i\mathbf{p} \cdot \mathbf{x}}}{K(\mathbf{p}) + \lambda_V}, \quad (70)$$

and compute its asymptotic behavior for  $L, M, |\mathbf{x}| \rightarrow \infty$  and  $\lambda_V \rightarrow 0$ . Using the Poisson summation formula (55) we obtain

$$G_V(\mathbf{x}) = \int_0^\infty dt e^{-\lambda_V t} \sum_n \int_{[-\pi, \pi]^d} \frac{d^d \mathbf{p}}{(2\pi)^d} e^{i\mathbf{p} \cdot (\mathbf{n}_\parallel M + \mathbf{x}_\parallel) + i\mathbf{p}_\perp \cdot (\mathbf{n}_\perp L + \mathbf{x}_\perp) - tK(\mathbf{p})}. \quad (71)$$

Since we are interested in the limit  $|\mathbf{x}|, L, M \rightarrow \infty$  we can replace  $K(\mathbf{p})$  with its small- $\mathbf{p}$  expansion and extend the integration to the whole space, obtaining

$$G_V(\mathbf{x}) \approx \frac{1}{(4\pi)^{d/2}} \int_0^\infty d\eta d\tau f_\rho(\eta) f_\sigma(\tau) \times \int_0^\infty dt e^{-\lambda_V t} \tau_t^{-q/2} \eta_t^{-p/2} \times \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{-|\mathbf{x}_\perp + L\mathbf{n}_\perp|^2 / (4\eta t) - |\mathbf{x}_\parallel + M\mathbf{n}_\parallel|^2 / (4\tau t)}. \quad (72)$$

By rescaling  $t$ , we obtain finally

$$G_V(\mathbf{x}) \approx \frac{L^{\rho(2-D)}}{(4\pi)^{d/2}} \int_0^\infty d\eta d\tau f_\rho(\eta) f_\sigma(\tau) \times \int_0^\infty dt e^{-(\lambda_V L^{2\rho}) t} \tau_t^{-q/2} \eta_t^{-p/2} \times \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{-|\mathbf{x}_\perp / L + \mathbf{n}_\perp|^2 / (4\eta t) - S^2 |\mathbf{x}_\parallel / M + \mathbf{n}_\parallel|^2 / (4\tau t)} = L^{\rho(2-D)} F(\mathbf{x}_\perp / L, \mathbf{x}_\parallel / M, S, \lambda_V L^{2\rho}) = \xi_\perp^{\rho(2-D)} \tilde{G}(\mathbf{x}_\perp / \xi_\perp, \mathbf{x}_\parallel / \xi_\parallel, S, \xi_\perp / L, S). \quad (73)$$

## Appendix C: Gap equation for $S \rightarrow \infty$

### C.1 General results

Now we will determine the limiting form of the gap equation (51) for  $L \rightarrow \infty, M \rightarrow \infty, S \rightarrow \infty$  and arbitrary

values of  $z \equiv (4\pi)^{-\rho/2} (L/\xi_{\perp,V})^\rho$ , including  $z = 0$ . Indeed, we shall mainly be interested in the limit  $z \rightarrow 0$  together with  $S \rightarrow \infty$ .

For this purpose we need to compute the asymptotic behavior of  $H_{p,q}(\eta, \tau, z, r)$  (see Eq. (62)) for  $r \rightarrow \infty$ . We start from equation (65). For what concerns the first term in this equation, we can take the limit naively, *i.e.* we can replace  $B(r\tau_t^{-1})$  with 1. The second term can be rewritten in the form

$$r^{-q/2} \int_1^\infty dt e^{-z^2 t} [B^p(\eta t) B^q(\tau_t/r) - 1] = r^{-q/2} \int_1^\infty dt e^{-z^2 t} [B^p(\eta t) - 1] B^q(\tau_t/r) + r^{-q/2} \int_1^\infty dt e^{-z^2 t} B^q(\tau_t/r) - r^{-q/2} \int_1^\infty dt e^{-z^2 t}. \quad (74)$$

In the first term we can simply replace  $B(\tau_t/r)$  with its asymptotic behavior for  $\tau_t/r$  small, *i.e.*  $(\tau_t/r)^{-1/2}$ , while some additional manipulations are needed for the second one. Rescaling  $t = sr^\sigma$ , we rewrite it as

$$r^{-q/2} \int_1^\infty dt e^{-z^2 t} B^q(\tau_t/r) = r^{\sigma-q/2} \int_1^\infty ds e^{-z^2 r^\sigma s} [B^q(\tau_s) - 1] + r^{\sigma-q/2} \int_{r^{-\sigma}}^1 ds e^{-z^2 r^\sigma s} \tau_s^{-q/2} [B^q(\tau_s^{-1}) - 1] + r^{\sigma-q/2} \frac{e^{-z^2 r^\sigma}}{z^2 r^\sigma} + r^{\sigma-q/2} \tau^{-q/2} \int_{r^{-\sigma}}^1 ds e^{-z^2 r^\sigma s} s^{-q/2\sigma}. \quad (75)$$

In the second term, one can easily convince himself that it is safe to replace the lower integration limit  $r^{-\sigma}$  with 0. Collecting everything together, we obtain

$$H_{p,q}(\eta, \tau, z, r) = \tau^{-q/2} \int_0^1 dt e^{-z^2 t} t^{-q/2\sigma} \eta_t^{-p/2} [B^p(\eta t^{-1}) - 1] + \tau^{-q/2} \int_1^\infty dt e^{-z^2 t} t^{-q/2\sigma} [B^p(\eta t) - 1] + r^{\sigma-q/2} \int_1^\infty dt e^{-z^2 r^\sigma t} [B^q(\tau t) - 1] + r^{\sigma-q/2} \int_0^1 dt e^{-z^2 r^\sigma t} \tau_t^{-q/2} [B^q(\tau t^{-1}) - 1] + r^{\sigma-q/2} \frac{e^{-z^2 r^\sigma}}{z^2 r^\sigma} + \tau^{-q/2} \int_1^{r^\sigma} dt e^{-z^2 t} t^{-q/2\sigma} - \tau^{-q/2} \eta^{-p/2} \int_1^\infty dt t^{-D/2} e^{-z^2 t}. \quad (76)$$

Then, for  $D > 2$ , we can write the gap equation as

$$(4\pi)^{\rho(2-D)/2+d/2}(\beta - \beta_c)L^{\rho(D-2)} = \\ - (4\pi)^{d/2}A_{p,q,\rho,\sigma}z^{D-2} + r^{\sigma-q/2}\mathcal{J}_1(z^2r^\sigma) \\ + \mathcal{J}_2(z^2) + C_{\sigma,q}\int_1^{r^\sigma} dt e^{-z^2t}t^{-q/2\sigma}, \quad (77)$$

where

$$\mathcal{J}_1(x) \equiv \frac{e^{-x}}{x} + \mathcal{G}_{\sigma,q}^{(1)}(x), \quad (78)$$

$$\mathcal{J}_2(x) \equiv -C_{\sigma,q}C_{\rho,p}\int_1^\infty dt t^{-D/2}e^{-xt} + \mathcal{G}_{\rho,p,\sigma,q}^{(2)}(x), \quad (79)$$

and

$$C_{\alpha,\beta} \equiv \int_0^\infty dp p^{-\beta/2}f_\alpha(p) = \frac{1}{\alpha} \frac{\Gamma(\beta/2\alpha)}{\Gamma(\beta/2)},$$

$$\mathcal{G}_{\sigma,q}^{(1)}(x) \equiv \int_0^\infty d\tau f_\sigma(\tau)G_{\sigma,q}^{(1)}(\tau, x),$$

$$G_{\sigma,q}^{(1)}(\tau, x) \equiv \int_1^\infty dt e^{-xt} [B^q(\tau_t) - 1] \\ + \int_0^1 dt e^{-xt}\tau_t^{-q/2} [B^q(\tau_t^{-1}) - 1] \\ = \int_0^\infty dt e^{-xt}\tau_t^{-q/2} [B^q(\tau_t^{-1}) - 1] \\ + \tau^{-q/2} \int_1^\infty dt e^{-xt}t^{-q/2\sigma} - \frac{e^{-x}}{x},$$

$$\mathcal{G}_{\rho,p,\sigma,q}^{(2)}(x) \equiv C_{\sigma,q} \int_0^\infty d\eta f_\rho(\eta)G_{\rho,p,\sigma,q}^{(2)}(\eta, x),$$

$$G_{\rho,p,\sigma,q}^{(2)}(\eta, x) \equiv \int_1^\infty dt e^{-xt}t^{-q/2\sigma} [B^p(\eta_t) - 1] \\ + \int_0^1 dt e^{-xt}t^{-q/2\sigma}\eta_t^{-p/2} [B^p(\eta_t^{-1}) - 1]. \quad (80)$$

Note that  $\mathcal{G}_{\sigma,q}^{(1)}(x)$ ,  $\mathcal{G}_{\rho,p,\sigma,q}^{(2)}(x)$ , and  $\mathcal{J}_2(x)$  are finite for  $x \rightarrow 0$  and the relation

$$\tau^{-q/2}G_{\rho,p,\sigma,q}^{(2)}(\eta, z^2) = H_{p,q}(\eta, \tau, z, \infty) \\ + \tau^{-q/2}\eta^{-p/2} \int_1^\infty dt e^{-z^2t}t^{-D/2} \\ - \tau^{-q/2} \int_1^\infty dt e^{-z^2t}t^{-q/2\sigma}. \quad (81)$$

We will be interested in computing the behavior of  $\mathcal{J}_1(x)$  for  $x \rightarrow \infty$ . If  $0 < \sigma \leq 1$ , we rewrite

$$\mathcal{J}_1(x) = C_{\sigma,q} \int_1^\infty dt e^{-xt}t^{-q/2\sigma} \\ + \int_0^\infty d\tau f_\sigma(\tau) \int_0^\infty dt e^{-xt}\tau_t^{-q/2} [B^q(\tau_t^{-1}) - 1]. \quad (82)$$

By performing an integration by parts it is easy to see that the first term vanishes as  $e^{-x}/x$ . The asymptotic behavior

of the last term depends on  $\sigma$ . For  $\sigma = 1$  we can rewrite it

$$\int_0^\infty dt e^{-xt}t^{-q/2} [B^q(t^{-1}) - 1] \\ \approx 2q \int_0^\infty dt t^{-q/2} e^{-xt - \pi t^{-1}} \\ = 2qx^{q/4-1/2} \int_0^\infty dt t^{-q/2} e^{-\sqrt{x}(t+\pi/t)} \\ \approx 2q \left(\frac{\pi}{x}\right)^{3/4-q/4} e^{-2\sqrt{\pi x}}. \quad (83)$$

For  $0 < \sigma < 1$ , more work is needed. First, we replace  $\tau$  with  $s = \tau t^{1/\sigma} \equiv \tau_t$  obtaining

$$\int_0^\infty dt t^{-1/\sigma} e^{-xt} \int_0^\infty ds \\ \times s^{-q/2} f_\sigma(st^{-1/\sigma}) [B^q(s^{-1}) - 1]. \quad (84)$$

In the limit of large  $x$ , only the small- $t$  region contributes to the integral (because of the exponential factor  $e^{-xt}$ ) and the function  $f_\sigma(st^{-1/\sigma})$  can be replaced with its small- $t$  behavior (this is safe because the integral over  $s$  is convergent near  $s = 0$ ), *i.e.* we can replace  $f_\sigma(x)$  with  $x^{-1-\sigma}$ . Thus, neglecting a multiplicative constant, the integral (84) becomes

$$\int_0^\infty dt t e^{-xt} \int_0^\infty ds s^{-1-\sigma-q/2} [B^q(s^{-1}) - 1] = \\ \frac{1}{x^2} \int_0^\infty ds s^{-1-\sigma-q/2} [B^q(s^{-1}) - 1], \quad (85)$$

and it is easy to check that for  $\sigma > 0$ ,  $q > 0$  the second integral is convergent. Therefore, as  $x \rightarrow \infty$ , we have that  $\mathcal{J}_1(x) \sim x^{-2}$  for any  $0 < \sigma < 1$  and  $\mathcal{J}_1(x) \sim x^p \exp(-C\sqrt{x})$  for  $\sigma = 1$ .

From equation (77) we see that there are two interesting scaling limits: (a)  $r \rightarrow \infty$  at fixed  $z$ ; (b)  $r \rightarrow \infty$  at fixed  $z^2r^\sigma$ . As we will show below these two limits correspond to the two different cases we discussed in Section 4. They are presented in detail below.

## C.2 Limit (a)

In this case we should consider the limit  $r \rightarrow \infty$  at fixed  $z$  and we should therefore reobtain equation (19) with  $S = \infty$ . This follows immediately from the fact that  $r^{\sigma-q/2}\mathcal{J}_1(z^2r^\sigma) \rightarrow 0$ , *cf.* Section 6, and from

$$\mathcal{J}_2(z^2) + C_{\sigma,q} \int_1^\infty dt e^{-z^2t}t^{-q/2\sigma} = \\ \int_0^\infty d\eta d\tau f_\rho(\eta)f_\sigma(\tau)H_{p,q}(\eta, \tau, z, \infty), \quad (86)$$

which is a direct consequence of equation (81).

### C.3 Limit (b)

Now we discuss the limit  $z^2 \rightarrow 0$  and  $r \rightarrow \infty$  at constant  $z^2 r^\sigma$ . Note that the last term in equation (77) behaves differently, depending on whether  $q/2\sigma$  is larger or smaller than 1. For  $q/\sigma > 2$ , integrating by parts, we obtain

$$\int_1^{r^\sigma} dt e^{-z^2 t} t^{-q/2\sigma} = \frac{2\sigma}{q-2\sigma} e^{-z^2} - \frac{2\sigma}{q-2\sigma} r^{\sigma-q/2} \times \left[ e^{-z^2 r^\sigma} + z^2 r^\sigma \int_{r^\sigma}^1 dt e^{-z^2 t} t^{1-q/2\sigma} \right]. \quad (87)$$

Then, note that for  $D < 4$  we can extend the remaining integration down to zero. Moreover, we can approximate  $e^{-z^2}$  with 1 since  $z^2$  is subleading with respect to  $r^{\sigma-q/2}$ : indeed  $z^2 \sim r^{-\sigma} \sim r^{\sigma-q/2} \times r^{q/2-2\sigma}$  and  $q < 4\sigma$  for  $D < 4$ . Therefore, if we define

$$K \equiv \mathcal{J}_2(0) + \frac{2\sigma}{q-2\sigma} C_{\sigma,q}, \quad (88)$$

$$\begin{aligned} \widehat{\mathcal{J}}_1(x) &\equiv \mathcal{J}_1(x) - \frac{2\sigma}{q-2\sigma} C_{\sigma,q} \\ &\times \left[ e^{-x} + x \int_0^1 dt e^{-xt} t^{1-q/2\sigma} \right] \\ &= \int_0^\infty d\tau dt f_\sigma(\tau) e^{-x\tau} \tau_t^{-q/2} [B^q(\tau_t^{-1}) - 1] \\ &+ C_{\sigma,q} \Gamma(1 - q/2\sigma) x^{(q-2\sigma)/2\sigma}, \end{aligned} \quad (89)$$

we obtain

$$(4\pi)^{\rho+(d-\rho D)/2} (\beta - \beta_c) L^{\rho(D-2)} = r^{\sigma-q/2} \widehat{\mathcal{J}}_1(z^2 r^\sigma) + K, \quad (90)$$

where we have also discarded the term proportional to  $z^{D-2} \sim (z r^{\sigma/2})^{D-2} r^{\sigma-q/2} r^{-\sigma p/2\rho}$ , which is subleading with respect to  $r^{\sigma-q/2}$ .

For  $q/\sigma < 2$ , we write

$$\begin{aligned} \int_1^{r^\sigma} e^{-z^2 t} t^{-q/2\sigma} dt &= r^{\sigma-q/2} \int_{r^\sigma}^1 e^{-z^2 r^\sigma t} t^{-q/2\sigma} dt \\ &= r^{\sigma-q/2} \int_0^1 e^{-z^2 r^\sigma t} t^{-q/2\sigma} dt \\ &- \int_0^1 e^{-z^2 t} t^{-q/2\sigma} dt. \end{aligned} \quad (91)$$

It is then easy to see that we reobtain equation (90). In this case  $K$  is subleading and can be neglected in the scaling limit.

In this derivation we have implicitly assumed that it is possible to take the limit  $z \rightarrow 0$ ,  $r \rightarrow \infty$ , with  $z^2 r^\sigma$  fixed. However, we now show that this is not the case for  $\beta < \beta_c$ . Indeed, the function  $\widehat{\mathcal{J}}_1(z)$  is positive and thus equation (90) can only be valid if  $\beta > \beta_c$ . If this condition is not satisfied, the only possibility is that  $(\beta - \beta_c) L^{\rho(D-2)}$  vanishes in the scaling limit as well as the right-hand side,

which in turn implies  $z^2 r^\sigma \rightarrow \infty$ . This means  $\lambda_V M^{2\sigma} \rightarrow \infty$ , or  $\chi/M^{2\sigma} \rightarrow 0$ .

Finally, we rewrite the gap equation in terms of the constant  $A_{q,\sigma} \equiv A_{0,q,\rho,\sigma}$  (it is easy to check that it is  $\rho$ -independent) and of the integral  $I_{q,\sigma}^{\text{iso}}(z)$  that appear in the gap equation of an isotropic  $q$ -dimensional system. It is indeed easy to show that

$$\widehat{\mathcal{J}}_1(z^2) = (4\pi)^{q/2} z^{-2+q/\sigma} [I_{q,\sigma}^{\text{iso}}(z) - A_{q,\sigma}]. \quad (92)$$

## References

1. M.E. Fisher, in *Critical Phenomena*, International School of Physics ‘‘Enrico Fermi’’, Course LI, edited by M.S. Green (Academic, New York, 1971)
2. M.N. Barber, *Finite-Size Scaling*, in *Phase Transitions and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz, Vol. 8 (Academic, London, 1983)
3. *Finite-Size Scaling*, edited by J.L. Cardy (North-Holland, Amsterdam, 1988)
4. V. Privman, *Finite Size Scaling and Numerical Simulation of Statistical Systems* (World Scientific, Singapore, 1990)
5. J.G. Brankov, D.M. Danchev, N.S. Tonchev, *Theory of Critical Phenomena in Finite-Size Systems* (World Scientific, Singapore, 2000)
6. K. Binder, Z. Phys. B **43**, 119 (1981)
7. M. Lüscher, P. Weisz, U. Wolff, Nucl. Phys. B **359**, 221 (1991)
8. H.W.J. Blöte, E. Luijten, J.R. Heringa, J. Phys. A **28**, 6289 (1995)
9. S. Caracciolo, R.G. Edwards, S.J. Ferreira, A. Pelissetto, A.D. Sokal, Phys. Rev. Lett. **74**, 2969 (1995)
10. S. Caracciolo, R.G. Edwards, A. Pelissetto, A.D. Sokal, Phys. Rev. Lett. **75**, 1891 (1995)
11. M. Hasenbusch, K. Pinn, S. Vinti, Phys. Rev. B **59**, 11471 (1999)
12. H.G. Ballesteros, L.A. Fernández, V. Martín-Mayor, A. Muñoz Sudupe, G. Parisi, J.J. Ruiz-Lorenzo, J. Phys. A **32**, 1 (1999)
13. H.W.J. Blöte, L.N. Shchur, A.L. Talapov, Int. J. Mod. Phys. C **10**, 1137 (1999)
14. M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, E. Vicari, Phys. Rev. B **63**, 214503 (2001); Phys. Rev. B **65**, 144520 (2002)
15. B. Schmittmann, R.K.P. Zia, *Statistical Mechanics of Driven Diffusive Systems*, in *Phase Transitions and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz, Vol. 17 (Academic, London, 1995)
16. J. Marro, R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1999)
17. J. Krug, Adv. Phys. **46**, 139 (1997)
18. R.M. Hornreich, J. Magn. Magn. Mater. **15-18**, 387 (1980)
19. W. Selke, in *Phase Transitions and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz, Vol. 15 (Academic, London, 1992)
20. M. Pleimling, M. Henkel, Phys. Rev. Lett. **87**, 125702 (2001)
21. H.W. Diehl, Acta Phys. Slov. **52**, 271 (2002)

22. A. Aharony, M.E. Fisher, Phys. Rev. B **8**, 3323 (1973); A. Aharony, Phys. Rev. B **8**, 3363 (1973)
23. A. Hucht, J. Phys. A **35**, L481 (2002)
24. E. Brézin, J. Zinn-Justin, Phys. Rev. B **13**, 251 (1976)
25. S.L. Sondhi, S.M. Girvin, J.P. Carini, D. Shahar, Rev. Mod. Phys. **69**, 315 (1997)
26. M. Henkel, Nucl. Phys. B **641**, 405 (2002); *Conformal Invariance and Critical Phenomena*, in *Texts and Monographs in Physics* (Springer-Verlag, Berlin, 1999)
27. S.M. Bhattacharjee, J.F. Nagle, Phys. Rev. A **31**, 3199 (1985)
28. K. Binder, J.S. Wang, J. Stat. Phys. **55**, 87 (1989)
29. K.-t. Leung, Phys. Rev. Lett. **66**, 453 (1991)
30. K.-t. Leung, Int. J. Mod. Phys. C **3**, 367 (1992)
31. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3d edn. (Oxford Science Publication, Clarendon Press, Oxford, 1996)
32. M.N. Barber, M.E. Fisher, Ann. Phys. [NY] **77**, 1 (1973)
33. E. Brézin, J. Physique **43**, 15 (1982)
34. M.E. Fisher, V. Privman, Commun. Math. Phys. **103**, 527 (1986)
35. J.G. Brankov, N.S. Tonchev, J. Stat. Phys. **52**, 143 (1988)
36. S. Singh, R.K. Pathria, Phys. Rev. B **40**, 9238 (1989)
37. S. Caracciolo, A. Gambassi, M. Gubinelli, A. Pelissetto, Eur. Phys. J. B **20**, 255 (2001)
38. H. Chamati, D.M. Dantchev, N.S. Tonchev, Eur. Phys. J. B **14**, 307 (2000)
39. J.M. Borwein, I.J. Zucker, IMA J. Numer. Anal. **12**, 519 (1992)
40. A. Cucchieri, T. Mendes, A. Pelissetto, A.D. Sokal, J. Stat. Phys. **86**, 581 (1997)
41. S. Caracciolo, A. Gambassi, M. Gubinelli, A. Pelissetto, J. Phys. A **36**, L315 (2003)